

## ALGEBRAS OF FOURIER TRANSFORMS WITH CLOSED RESTRICTIONS

BY

BENJAMIN B. WELLS, JR.

**ABSTRACT.** Let  $G$  denote a compact abelian group and let  $B$  denote a Banach subalgebra of  $A$ , the algebra of complex-valued functions on  $G$  whose Fourier series is absolutely convergent. If  $B$  contains the constant functions, separates the points of  $G$ , and if the restriction algebra,  $B(E)$ , is closed in  $A(E)$  for every closed subset  $E$  of  $G$ , then  $B = A$ .

**1. Introduction.** Let  $G$  denote a compact abelian group and  $A$  the algebra under the pointwise operations of complex-valued functions on  $G$  having absolutely convergent Fourier series. Thus, a function  $f$  in  $A$  has a representation as  $\sum a_n(\gamma_n, g)$  where  $\sum |a_n| < \infty$ , the latter quantity serving as the norm of  $f$ .

Let  $B$  denote a subalgebra continuously embedded in  $A$ , containing the constant functions, and separating the points of  $G$ . For  $E$ , a closed subset of  $G$ ,  $B(E)$  and  $A(E)$  denote the restriction algebras to the set  $E$  and are furnished with the quotient norm.

**THEOREM.** *Suppose that for every closed set  $E \subset G$ ,  $B(E)$  is a closed linear subspace of  $A(E)$ . Then  $B = A$ .*

This result is the exact analogue of a result [1] in function algebras with the role of the continuous complex-valued functions played by  $A$ . In his thesis [4] Sungwoo Suh proved the theorem for locally compact totally disconnected groups.

To illustrate the content of the Theorem, consider the case of  $G$  equal to  $T$ , the circle group, and let  $B$  be the closure of the linear span of  $A$ -functions taking the values 1 and 0 on a closed totally disconnected set  $F \subset T$ . It is known (see [2, p. 40], and [3]) that there are certain  $F$  for which  $B$  is a proper subalgebra of  $A$ . From our Theorem it follows that  $B(E)$  fails to be closed in  $A(E)$  for some closed  $E$  that must intersect  $F$  in a proper subset. A qualitative statement of this fact is the following. There are functions locally constant on  $E$  that are the restrictions of  $A$ -functions having small  $A$ -norm, but such that any attempt to extend them to remain locally constant on  $F$  will force their  $A$ -norms to become large.

We now indicate some notations and tools to be used in the proof of the Theorem. The norm of a function  $f \in B(E)$  is given by

$$\|f\|_{B(E)} = \inf\{\|f + h\|_B : h \in B \text{ and } h = 0 \text{ on } E\}.$$

If  $f \in B(E)$ , a *representative* of  $f$  is a function  $f_0$  belonging to  $B$  such that  $f_0|_E = f$ .

---

Received by the editors April 10, 1979 and, in revised form, December 10, 1979.

AMS (MOS) subject classifications (1970). Primary 43A25, 46E25.

© 1980 American Mathematical Society  
0002-9947/80/0000-0371/\$02.50

The dual space of  $A$  as a Banach space is  $PM$ , the space of pseudomeasures on  $G$ . The action of a pseudomeasure  $S$  on an  $A$ -function  $f$  will be denoted by  $(S, f)$ . The symbol  $Sf$  will denote that pseudomeasure such that  $(Sf, g) = (S, fg)$  for all  $g \in A$ . A pseudomeasure  $S$  is said to be supported by  $E$ , and we write  $S \in PM(E)$ , provided  $(S, f) = 0$  whenever  $f$  vanishes in a neighborhood of  $E$ . The dual space of the restriction algebra  $A(E)$  consists of all those pseudomeasures belonging to  $PM(E)$  that annihilate every  $A$ -function vanishing on  $E$ . Finally,  $B^\perp$  will denote all of those pseudomeasures that annihilate every function in  $B$ .

**2. Preliminary lemmas.** We begin by showing that under the assumptions of the Theorem the restriction algebra  $B(E)$  must contain all of the idempotents of  $A(E)$ . In the function algebra case (cf. [1]) this step is almost immediate. In this section  $G$  need not be a group, only a compact topological space. The algebra  $A$  need only be a commutative semisimple regular Banach algebra.

Let  $A$  be such an algebra under the pointwise operations on its maximal ideal space  $G$ , and let  $B$  denote a closed subalgebra of  $A$  containing 1 and separating the points of  $G$ .

**LEMMA 1.** *Assume that for every closed subset  $E \subset G$  the inclusion mapping  $B(E) \rightarrow A(E)$  is closed. Then, for every closed subset  $E \subset G$ ,  $B(E)$  contains all the idempotent functions on  $E$  belonging to  $A(E)$ .*

**PROOF.** Let  $E$  be an arbitrary closed subset of  $G$ . We will show that the maximal ideal space of  $B(E)$  is  $E$ . Thus, appealing to the Shilov idempotent theorem, we have that if  $f$  belongs to  $A(E)$  and is idempotent on  $E$ , then it must also belong to  $B(E)$ .

Let  $h$  denote a complex homomorphism of  $B(E)$ , and suppose that  $h$  is not evaluation at some point of  $E$ . Then for each  $x \in E$ , there exists a function  $g$  belonging to  $B(E)$  such that  $h(g) = 1$  and such that  $g(x) = 0$ . Let  $W_x$  be a closed neighborhood of  $x$  such that  $|g(y)| \leq 1/4$  for all  $y \in W_x$ . Of course, the maximal ideal space of  $B(E \cap W_x)$ ,  $\Delta$ , is a closed subset of the maximal ideal space of  $B(E)$ . We shall now show that  $h$  does not belong to  $\Delta$ .

Suppose on the contrary that  $h \in \Delta$ , and define  $a = \sup_{\phi \in \Delta} |\phi(g)|$ . Then  $a \geq 1$ , and for every positive integer  $n$

$$\|(g/a)^n\|_{B(W_x \cap E)} \geq 1. \quad (1)$$

Since  $|g(y)| \leq 1/4$  for all  $y \in W_x$ , and the maximal ideal space of  $A(W_x \cap E)$  is just  $W_x \cap E$ , it follows that

$$\limsup \| (g/a)^n \|_{A(W_x \cap E)}^{1/n} \leq 1/4.$$

Thus, for  $n$  large we have

$$\|(g/a)^n\|_{A(W_x \cap E)} \leq (1/2)^n. \quad (2)$$

Now, (1) and (2) together contradict the closedness of the mapping  $B(W_x \cap E) \rightarrow A(W_x \cap E)$ . This establishes that  $h$  does not belong to  $\Delta$ .

Therefore, there is a function  $g \in B(E)$  that vanishes on  $W_x$  such that  $h(g) = 1$ . We may repeat the above argument for each point of  $E$ , and by the compactness of

$E$  choose a finite set of functions  $g_i, i = 1, \dots, n$ , so that  $h(g_i) = 1$  and so that the product  $g_0$  vanishes on  $E$ . Of course,  $h(g_0) = 1$ . This contradicts the fact that  $h$  was chosen in the maximal ideal space of  $B(E)$ . The proof of the lemma is complete.

DEFINITION. Let  $N_E$  denote the norm of the inverse of the inclusion mapping  $B(E) \rightarrow A(E)$ . Following [1], we say that  $B$  is *bounded on a subset*  $V$  of  $G$  if there is a positive constant  $C_V$  for which, whenever  $F$  is closed and contained in  $V$ ,  $N_F \leq C_V$ . We say that  $B$  is bounded at  $x \in G$  if  $B$  is bounded on some neighborhood  $V$  of  $x$ .

LEMMA 2. Let  $V_1$  and  $V_2$  be open subsets of  $G$  and let  $B$  be bounded on each  $V_i, i = 1, 2$ . Then  $B$  is bounded on every closed subset of  $V_1 \cup V_2$ .

PROOF. Choose a closed subset  $F$  of  $V_1 \cup V_2$ . Since  $F \setminus V_2$  is a closed subset of  $V_1$ , we can find open  $W_1 \supset F \setminus V_2$  such that  $\overline{W_1} \subset V_1$ . Since  $F \setminus W_1 \subset V_2$ , there is an open  $W_2 \supset F \setminus W_1$  such that  $\overline{W_2} \subset V_2$ . Now  $F \subset W_1 \cup W_2$ . Furthermore, we may choose open  $L$  satisfying  $F \setminus W_2 \subset L \subset \overline{L} \subset W_1$ , and by Lemma 1 a function  $h \in B$  that is 1 on  $F \setminus W_2$  and 0 on  $L^c$ . For  $f \in B(F)$  there are  $f_i \in B, i = 1, 2$ , such that  $f_i|_{\overline{W_i}} = f|_{\overline{W_i}}$  and  $\|f_i\|_B < C_{V_i}$ . But now it is easy to check that  $f = f_1 \cdot h + (1 - h)f_2$  on  $F$ . We have, therefore

$$\|f\|_{B(F)} \leq \|f_1\|_B \|h\|_B + \|1 - h\|_B \|f_2\|_B < \|h\|_B C_{V_1} + \|1 - h\|_B C_{V_2}. \quad (3)$$

The right-hand side of (3) may be taken to be  $C_F$ . The proof of the lemma is now complete.

LEMMA 3. If  $F$  is closed in  $G$ , and  $B$  is bounded at each  $x \in F$ , there is an open  $V$  containing  $F$  on which  $B$  is bounded.

PROOF. The proof is an obvious consequence of compactness of  $F$  and Lemma 2.

LEMMA 4. There are at most finitely many  $x \in G$  at which  $B$  is not bounded.

PROOF. Assume on the contrary that  $B$  is not bounded at each member of the sequence  $\{y_n\}$ . By dropping to a subsequence, if necessary, we may suppose that no element of the sequence  $\{y_n\}$  is a limit point of the sequence. Choose open neighborhoods  $U_{y_1}, W_{y_1}$  of  $y_1$  that are disjoint from the set of limit points of the sequence  $\{y_n\}$ , such that  $\overline{W_{y_1}} \subset U_{y_1}$ . By Lemma 1 we may choose a function  $k_1 \in B$  that is 1 on  $\overline{W_{y_1}}$  and 0 on  $U_{y_1}^c$ . Since  $B$  is unbounded in every neighborhood of  $y_1$ , we may pick closed  $F_1 \subset W_{y_1}$  so that there is a  $g_1 \in B(F_1)$  with  $\|g_1\|_{B(F_1)} > 1$  while having  $\|g_1\|_{A(F_1)} < \|k_1\|_A^{-1}$ . Set  $f_1 = h_1 k_1$  where  $h_1|_{F_1} = g_1$  and  $h_1$  is a representative of  $g_1$  whose  $A$ -norm is less than  $\|k_1\|_A^{-1}$ . Then  $\|f_1\|_A \leq \|h_1\|_A \|k_1\|_A < 1$ , but, of course,  $\|f_1\|_{B(F_1)} > 1$  since  $f_1 = g_1$  on  $F_1$ . Note that  $f_1$  itself may not belong to  $B$ , since  $h_1$  need not.

Next choose  $y'_2 \in \{y_n\}_{n=2}^\infty$  and open neighborhoods  $U_{y'_2}, W_{y'_2}$  of  $y'_2$  that are disjoint from the set of limit points of the sequence  $\{y_n\}$ , such that  $\overline{W_{y'_2}} \subset U_{y'_2}$  and  $U_{y_1} \cap U_{y'_2} = \emptyset$ . Choose a function  $k_2 \in B$  such that  $k_2$  is 1 on  $\overline{W_{y'_2}}$  and is 0 on  $U_{y'_2}^c$ . Since  $B$  is not bounded in every neighborhood of  $y'_2$ , it follows that there is a closed set  $F_2 \subset W_{y'_2}$  and a function  $g_2 \in B(F_2)$  with  $\|g_2\|_{B(F_2)} > 2$ , but  $\|g_2\|_{A(F_2)} < \|k_2\|_A^{-1}$ .

Set  $f_2 = h_2 \cdot k_2$  where  $h_2|_{F_2} = g_2$  and  $h_2$  is a representative of  $g_2$  whose  $A$ -norm is less than  $\|k_2\|_A^{-1}$ . Then  $\|f_2\|_A < 1$ , but, of course,  $\|f_2\|_{B(F_2)} > 2$ . Continuing in this way, we arrive at a disjoint sequence of closed sets  $\{F_n\}$  and a sequence of functions  $\{f_n\}$  such that

$$\|f_n\|_A < 1, \quad \|f_n\|_{B(F_n)} > n, \quad n = 1, 2, \dots \quad (4)$$

Denote by  $F$  the closed set  $\bigcup_{n=1}^{\infty} F_n \cup F_0$ , where  $F_0$  is the set of limit points of  $\bigcup_{n=1}^{\infty} F_n$ . For each  $m$ ,  $f_m|_{F_m} \in B(F_m)$  and  $f_m = 0$  on the closed set  $\bigcup_{n \neq m} F_n \cup F_0$ . The latter set is disjoint from  $F_m$ . It follows from Lemma 1, by multiplying any representative of  $f_m|_{F_m}$  in  $B$  by a function in  $B$  equal to 1 on  $F_m$  and 0 on  $\bigcup_{n \neq m} F_n \cup F_0$ , that  $f_m \in B(F)$ . Therefore, the closedness of the inclusion mapping  $B(F) \rightarrow A(F)$  is contradicted by (4). The proof of the lemma is complete.

**3. Proof of the Theorem for the circle  $T$ .** Assume on the contrary that  $B \neq A$  and that  $H$  is the finite set of points at which  $B$  is not bounded. Our first claim is that there is a closed set  $F$  such that  $F^c$  contains  $H$  and such that  $B(F) \neq A(F)$ . If not, then for every such  $F$ ,  $B(F) = A(F)$ . Let  $S$  denote a pseudomeasure in  $B^\perp$ , and suppose that  $f$  belongs to  $A$  and is 0 in a neighborhood of  $H$ . It follows from Lemma 1 that  $B$  is a normal algebra of functions on  $T$ . Therefore, since  $f$  belongs locally to  $B$  at each point of  $T$ ,  $f$  belongs to  $B$ . Since  $S$  is orthogonal to  $B$ ,  $(S, f) = 0$ . Thus the support of  $S$  is the finite set  $H$ , and  $S$  is in fact a measure. Since  $B$  separates the points of  $T$ , it follows that  $S\{p\} = 0$  for every  $p \in H$ . This establishes that  $B(F) \neq A(F)$  for some closed set  $F$  such that  $F^c$  contains  $H$ . Henceforth we shall assume that  $F^c$  is a finite union of disjoint open intervals centered at the points of  $H$ . Then, by Lemma 4,  $B$  is bounded, say with constant  $N_W$ , on some open set  $W$  containing  $F$  and missing  $H$ .

Now, at least one of the functions  $e^{\pm ix}$ , say  $e^{ix}$ , when restricted to  $F$ , fails to belong to  $B(F)$ . By the Hahn-Banach Theorem there is a nonzero element  $S$  of  $A(F)^*$  that annihilates  $B(F)$  and such that  $(S, e^{ix}) = 1$ . We shall regard  $S$  as a pseudomeasure supported on  $F$ .

Let  $m$  be a fixed positive integer satisfying

$$\exp(20/m) < 1 + (8e^{20}\|S\|N_W^2C)^{-1}, \quad (5)$$

where  $C$  denotes the norm of the mapping  $B \rightarrow A$ .

Let  $\sigma$  denote the trapezoidal function equal to 1 on the interval  $(-\pi/4, \pi/4)$ , zero outside the interval  $(-\pi/2, \pi/2)$  and linear on intermediate intervals. Set  $V_1(x) = \sigma(mx)$  and  $V_2 = 1 - V_1$ . From standard estimates it follows that  $\|V_j\|_A \leq 4$  for  $j = 1, 2$ . By Lemma 1 and the boundedness of  $B$  on  $W$ , there is a  $B$ -function  $v_1$  with its  $B$ -norm bounded by  $4N_W$ , and such that  $v_1 = V_1$  on the intersection of  $F$  with the intervals where  $V_1$  is constant. Set  $v_2 = 1 - v_1$ .

Since  $(S, e^{ix}) = 1$ , it follows that for at least one  $j$ ,  $|(Sv_j, e^{ix})| \geq 1/2$ . There will be no loss of generality in assuming that this occurs for  $j = 1$ . The ensuing argument is similar in the other case. Let  $K$  denote the support of the function  $v_1$ . Of course, since  $S$  is orthogonal to  $B$ , so also is the pseudomeasure  $L = Sv_1$ . The latter's support is contained in the finite union of closed intervals  $F \cap K$ , and its

pseudomeasure norm satisfies

$$\|L\| \leq \|S\| \|v_1\|_A \leq 4\|S\| N_W C. \quad (6)$$

Let  $\zeta(x)$  denote the continuous piecewise linear function such that  $\zeta(-\pi/2) = 0$ ,  $\zeta(\pi/2) = \pi$  and  $\zeta$  has slope  $-1$  on the intervals complementary to  $(-\pi/2, \pi/2)$ . Set  $Z(x) = \zeta(mx)$ . By standard estimates the  $A$ -norm of  $Z$  is no greater than 20. The important property for present purposes of the function  $Z$  is that the restriction of the function  $e^{iZ/m}$  to the support of  $v_1$  is just  $\exp\{i(x + \lambda_j)\}$  for constants  $\lambda_j$  and  $x$  belonging to the  $j$ th interval of support of  $v_1$ ,  $j = 1, 2, \dots, m$ .

The  $A$ -norm of the function  $e^{i(x-Z/m)}$  is no greater than  $e^{20}$ . On the set  $F \cap K$  it is equal to the constant  $e^{-i\lambda_j}$  for  $x$  belonging to the  $j$ th interval of  $K$ . In particular, it is locally constant on  $F \cap K$  and so belongs to  $B(F \cap K)$ . Its  $B(F \cap K)$ -norm is no greater than  $N_W e^{20}$ . Let  $\alpha(x)$  be a representative in  $B$  of the restriction of  $e^{i(x-Z/m)}$  to the set  $F \cap K$  satisfying

$$\|\alpha\|_B \leq N_W e^{20}. \quad (7)$$

Likewise, let  $\alpha_1(x)$  be a representative in  $B$  of  $e^{-i(x-Z/m)}$  restricted to  $F \cap K$ . Thus, we have that

$$1/2 \leq |(L\alpha\alpha_1, e^{ix})| = |(L\alpha, \alpha_1 e^{ix})| = |(L\alpha, e^{iZ/m})|. \quad (8)$$

Since  $\alpha \in B$ , the pseudomeasure  $L\alpha$  is still orthogonal to  $B$ .

We have

$$\|e^{iZ/m} - 1\|_{A(F \cap K)} \leq \exp\{\|Z\|_A/m\} - 1. \quad (9)$$

From the triangle inequality we see that

$$|(L\alpha, e^{iZ/m})| \leq |(L\alpha, e^{iZ/m} - 1)| + |(L\alpha, 1)|. \quad (10)$$

Since  $L\alpha$  is orthogonal to  $B$ , the second term on the right-hand side of (10) is 0. Since the set  $F \cap K$  is a finite union of closed intervals,  $L\alpha$  may be regarded as an element of the dual space of  $A(F \cap K)$ . Therefore, the first term on the right-hand side of (10) is at most  $\|L\| \|\alpha\|_A \|e^{iZ/m} - 1\|_{A(F \cap K)}$ . By our choice of  $m$  in (5) and by inequalities (6), (7) and (9), it follows that the right-hand side of (10) is less than  $1/2$ . This contradicts (8) and completes the proof of the Theorem for the case of the circle.

**4. Proof of the Theorem for an arbitrary compact abelian group.** First we consider a product of circles  $\prod_\alpha T_\alpha$ . Let  $f$  denote the element in the dual group constant in every coordinate except the  $\beta$ th coordinate, and of the form  $e^{ix}$  in the  $\beta$ th coordinate. Consider the products  $E = \prod_{\alpha \neq \beta} T_\alpha \times E_\beta$ , where the projection onto the  $\beta$ th coordinate is a closed set  $E_\beta$ .

Since  $B(F)$  is closed in  $A(F)$  for all  $F$  closed and contained in  $\prod_\alpha T_\alpha$ , it follows from Lemma 1 that  $B(E)$  contains the span of the idempotents of  $A(E)$ . By Lemma 4  $B$  is bounded on the complement of any neighborhood of a certain finite set of points. To deny that  $f$  belongs to  $B$  leads to a contradiction as before. Therefore, all characters having the form of  $f$  belong to  $B$ , and since  $B$  is closed in  $A$  it must equal  $A$ .

Finally, let  $G$  be an arbitrary compact abelian group. Then every neighborhood of the identity contains a closed subgroup  $H$  such that  $G/H$  is isomorphic to a (finite) product of circles with a finite group. By the above remarks and Lemma 1,  $B$  will contain those characters of the dual group  $G/H$ . If  $B(G/H)$  is interpreted to mean those  $B$ -functions constant on the cosets of  $H$ , it follows that  $B(G/H) = A(G/H)$ . In particular,  $B$  contains all trigonometric polynomials, and by the closedness of  $B$  in  $A$ , we again conclude that  $B = A$ .

#### REFERENCES

1. I. Glicksberg, *Function algebras with closed restrictions*, Proc. Amer. Math. Soc. **14** (1963), 158–161.
2. J.-P. Kahane, *Séries de Fourier absolument convergentes*, Ergebnisse der Math. und ihrer Grenzgebiete, Band 50, Springer-Verlag, Berlin and New York, 1970.
3. Y. Katznelson and W. Rudin, *The Stone-Weierstrass property in Banach algebras*, Pacific J. Math. **11** (1961), 253–265.
4. Sungwoo Suh, *Characterization of  $L^1(G)$  among its subalgebras*, Thesis, Univ. of Connecticut, 1978.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAWAII AT MANOA, HONOLULU, HAWAII 96822